I. Review of Process Dynamics

A. Example: Mixing Tank

F \rightarrow \text{Tank} \rightarrow F
\text{C}_{Ao} \rightarrow V \rightarrow \text{C}_A

\text{Governing eqn.}

V \frac{d\text{C}_A}{dt} = F(\text{C}_{Ao} - \text{C}_A)

(1)

At \ t = 0, \ \text{C}_{Ao} = \text{C}_{Ao i} = \text{C}_A

For \ t > 0, \ \text{C}_{Ao} \text{ is stepped to a new constant value, } \text{C}_{Ao f}

Let \ \frac{V}{F} = \tau. \ \text{Eqn. (1) becomes}

\tau \frac{d\text{C}_A}{dt} = \text{C}_{Ao} - \text{C}_A

F (\text{m}^3/s)
\text{C} (\text{kg/m}^3)
At initial steady state

\[
\tau \frac{dC_{Ao}}{dt} = 0 = (C_{Ao} - C_{Ao_i})
\]  \hspace{1cm} (2)

Subtracting (2) from (1) gives

\[
\tau \frac{d(C_A - C_{Ao_i})}{dt} = [C_{AoF} - C_{Ao_i} - (C_A - C_{Ao})]
\]

Let \( C_A - C_{Ao_i} = \Delta A \)

\[
C_{AoF} - C_{Ao_i} = \Delta C_{Ao}
\]

\[
\tau \frac{dC_A'}{dt} = \Delta C_{Ao} - C_A' \hspace{1cm} (3)
\]

at \( t=0, \ C_A'(0) = 0 \)

Solution: separate variables and integrate

\[
\int_{C_A'}^{C_A'} \frac{dC_A'}{\Delta C_{Ao} - C_A'} = \frac{1}{\tau} \int_0^t dt
\]

\[
- \ln (\Delta C_{Ao} - C_A') \bigg|_0^{C_A'} = \frac{t}{\tau}
\]
\[ \frac{\mu (\Delta C_{A_0} - C_A')}{\Delta C_{A_0}} = -\frac{t}{\tau} \]

\[ \Rightarrow 1 - \frac{C_A'}{\Delta C_{A_0}} = e^{-t/\tau} \]

\[ C_A'(t) = \Delta C_{A_0} \left( 1 - e^{-t/\tau} \right) \] (4)

where \( \tau = \frac{V}{F} \)

Solution: by Laplace transforms

\[ \tau \frac{dC_A'}{dt} = \Delta C_{A_0} - C_A' \] (3)

at \( t = 0, \ C_A'(0) = 0 \)

Transform to Laplace domain

\[ \mathcal{L} \left\{ \Delta C_{A_0} \right\} = \frac{\Delta C_{A_0}}{s}, \quad \mathcal{L} \left\{ -C_A' \right\} = -C_A(s) \]

\[ \Rightarrow \mathcal{L} \left\{ C_A(s) \right\} = \frac{1}{\tau s + 1} \frac{\Delta C_{A_0}}{s} \]

\[ \Rightarrow C_A(s) = \frac{\Delta C_{A_0}}{s} \frac{1}{\tau s + 1} \]
\[ C_A(t) = \Delta C_{A_0} \left( 1 - e^{-t/\alpha} \right) \]  
\(4\)

B. Block Diagrams

\[ Y(s) = G(s) X(s) \]

for mixing tank

\[ C_A(s) = \frac{1}{s + 1} \frac{\Delta C_{A_0}}{3} \]

1.0 Block diagram algebra

\[ Y(s) = G(s) X(s) \]
\[ X_1(s) \rightarrow X_3(s) \]
\[ X_2(s) \rightarrow X_3(s) \]

\[ X_3(s) = X_1(s) + X_2(s) \]

\[ X_1(s) = X_2(s) = X_3(s) \]

C. Stability

1. A pole is a root of the denominator of the transfer function (based on partial fraction expansion).

For the given transfer function:

\[ G(s) = \frac{1}{\varepsilon s + 1} \]

There is one pole, \( s = -\frac{1}{\varepsilon} \).
2.0 If any real part of the pole is greater than zero, the system is unstable. If negative, then stable.

3.0 Application to Feedback Loop

For a feedback control system, for what values of $K_c$ is the system stable?

\[ G_p(s) = \frac{0.1}{(s+1)(2s+1)} \]

Overall transfer function

\[ C(s) = K_c G_p(s)(R(s) - C(s)) \]

\[ C(s) = K_c G_p(s) R(s) - K_c G_p(s) C(s) \]
\[ C(s) \left[ 1 + K_c G_p(s) \right] = K_c G_p(s) R(s) \]

\[ C(s) = \frac{K_c G_p(s) R(s)}{1 + K_c G_p(s)} \]

\[ G(s) = \frac{K_c G_p(s)}{1 + K_c G_p(s)} = \frac{C(s)}{R(s)} \]

The overall transfer function is

The characteristic eqn. is \( 1 + K_c G_p(s) = 0 \).

Stable if roots of characteristic eqn. have negative real parts.

Unstable if have any positive real parts.