Matrix & Vector Multiplication

ChEn 1703

Wiki page notes

THE UNIVERSITY OF UTAH

Matrix-Vector Multiplication

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \qquad c = A\,b = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

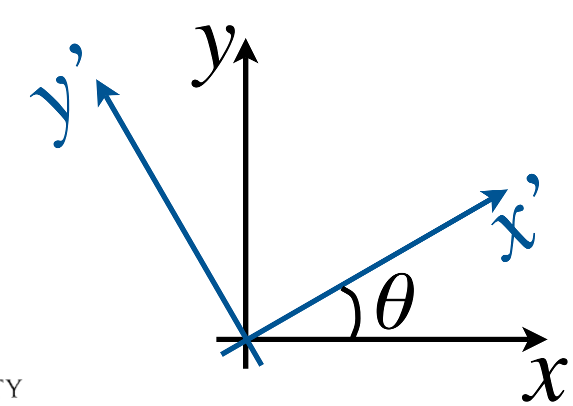
Rules:

1. b is a column vector with as many rows as there are columns in A
2. c is a column vector with as many rows as there are in A.

$$c_i = \sum_{j=1}^{n} A_{i,j}\, b_j$$

Example:

Given coordinates in (x,y), find coordinates in (x',y').



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}$$

$(x,y) = (1,1)$

$(x',y') = (?,?)$

Example:

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 3 & 2 & 10 \end{bmatrix} \qquad b = \begin{pmatrix} 7 \\ 1 \\ 4 \end{pmatrix}$$

Rules:

1. b is a column vector with as many rows as there are columns in A
2. c is a column vector with as many rows as there are in A.

Can they be multiplied (rule 1)?

$$c_i = \sum_{j=1}^{n} A_{i,j}\, b_j$$

What size is c (rule 2)? $c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 3 & 2 & 10 \end{bmatrix} \qquad b = \begin{pmatrix} 7 \\ 1 \\ 4 \end{pmatrix}$$

$$\begin{aligned} c_1 &= \sum_{j=1}^{3} A_{1,j} b_j \\ &= A_{1,1}b_1 + A_{1,2}b_2 + A_{1,3}b_3 \\ &= 1 \cdot 7 + 5 \cdot 1 + 2 \cdot 4 = \boxed{20} \end{aligned}$$

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 3 & 2 & 10 \end{bmatrix} \qquad b = \begin{pmatrix} 7 \\ 1 \\ 4 \end{pmatrix}$$

$$\begin{aligned} c_2 &= \sum_{j=1}^{3} A_{2,j} b_j \\ &= A_{2,1}b_1 + A_{2,2}b_2 + A_{2,3}b_3 \\ &= 3 \cdot 7 + 2 \cdot 1 + 10 \cdot 4 = \boxed{63} \end{aligned}$$

$$c = \begin{pmatrix} 20 \\ 63 \end{pmatrix}$$

Matrix-Matrix Multiplication

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & \cdots & b_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{n\ell} \end{bmatrix}$$

Matrix-matrix multiplication: $C_{ij} = \sum_{k=1}^{n} A_{i,k} B_{k,j}$

<u>Rules</u>:

1. B has as many rows as A has columns.
2. C has as many rows as A and as many columns as B.

Matlab command:

`C=A*B;`

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 4 & 10 \\ 6 & 3 & 1 \end{bmatrix}$

Rules:
1. B has as many rows as A has columns.
2. C has as many rows as A and as many columns as B.

Can they be multiplied (rule 1)?

What size is C (rule 2)? $C = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \end{bmatrix}$

$$C_{ij} = \sum_{k=1}^{n} A_{i,k} B_{k,j}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 4 & 10 \\ 6 & 3 & 1 \end{bmatrix}$$

$$\begin{aligned} c_{1,1} &= \sum_{k=1}^{2} A_{1,k} B_{k,1} \\ &= A_{1,1}B_{1,1} + A_{1,2}B_{2,1} \\ &= 1 \cdot 2 + 2 \cdot 6 = \boxed{14} \end{aligned}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 4 & 10 \\ 6 & 3 & 1 \end{bmatrix}$$

$$\begin{aligned} c_{1,2} &= \sum_{k=1}^{2} A_{1,k} B_{k,2} \\ &= A_{1,1}B_{1,2} + A_{1,2}B_{2,2} \\ &= 1 \cdot 4 + 2 \cdot 3 = \boxed{10} \end{aligned}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 4 & 10 \\ 6 & 3 & 1 \end{bmatrix}$$

$$\begin{aligned} c_{1,3} &= \sum_{k=1}^{2} A_{1,k} B_{k,3} \\ &= A_{1,1}B_{1,3} + A_{1,2}B_{2,3} \\ &= 1 \cdot 10 + 2 \cdot 1 = \boxed{12} \end{aligned}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 4 & 10 \\ 6 & 3 & 1 \end{bmatrix}$$

$$\begin{aligned} c_{2,1} &= \sum_{k=1}^{2} A_{2,k} B_{k,1} \\ &= A_{2,1}B_{1,1} + A_{2,2}B_{2,1} \\ &= 3 \cdot 2 + 5 \cdot 6 = \boxed{36} \end{aligned}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 4 & 10 \\ 6 & 3 & 1 \end{bmatrix}$$

$$\begin{aligned} c_{2,2} &= \sum_{k=1}^{2} A_{2,k} B_{k,2} \\ &= A_{2,1}B_{1,2} + A_{2,2}B_{2,2} \\ &= 3 \cdot 4 + 5 \cdot 3 = \boxed{27} \end{aligned}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 4 & 10 \\ 6 & 3 & 1 \end{bmatrix}$$

$$\begin{aligned} c_{2,3} &= \sum_{k=1}^{2} A_{2,k} B_{k,3} \\ &= A_{2,1}B_{1,3} + A_{2,2}B_{2,3} \\ &= 3 \cdot 10 + 5 \cdot 1 = \boxed{35} \end{aligned}$$

Vector Operations - Dot Product

- Dot Product (inner product, scalar product)

$$\vec{a} \bullet \vec{b} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \bullet \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = a_x b_x + a_y b_y + a_z b_z$$

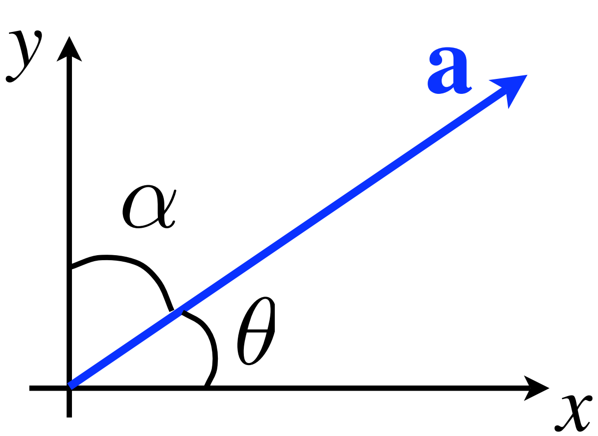
- `c=dot(a,b);`
- `c=sum(a.*b);`
- if a & b are column vectors:
  - `c=a'*b` is the dot product (Why does this work?)

A few other useful tidbits:

$$\vec{a} \bullet \vec{b} = |a|\,|b| \cos\theta$$

$$\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$$

$$|a| = \sqrt{\sum_{i=1}^{n} a_i^2}$$



$\hat{i}$ unit vector in x direction.

$\hat{j}$ unit vector in y direction.

$$\mathbf{a} \cdot \hat{i} = a_x \cdot 1 + a_y \cdot 0 = \boxed{a_x}$$

$$\mathbf{a} \cdot \hat{j} = a_x \cdot 0 + a_y \cdot 1 = \boxed{a_x}$$

$$a_x = |a||\hat{i}| \cos\theta$$

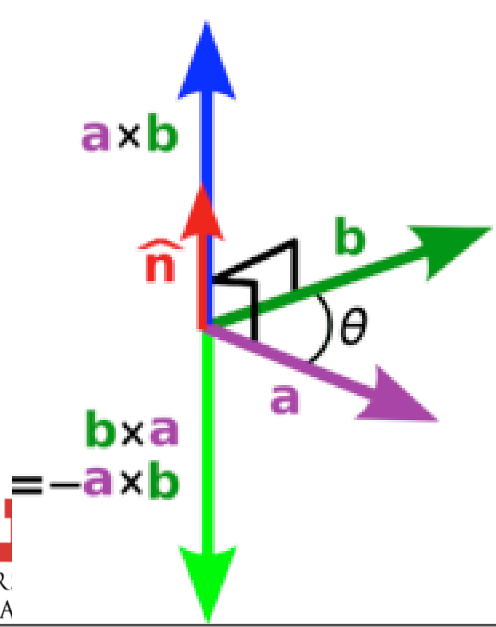
$$a_y = |a||\hat{j}| \cos\alpha$$

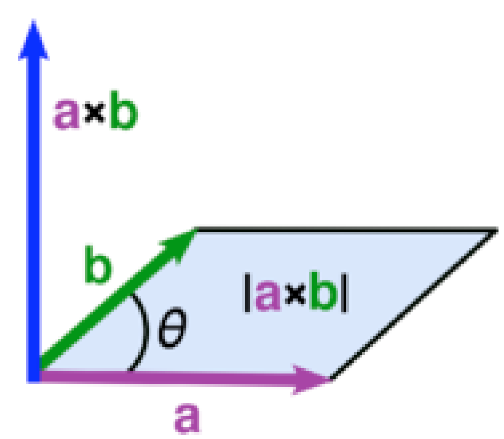
Vector Operations - Cross Product

- Cross product (outer product, vector product)

$$\vec{a} \times \vec{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \qquad \begin{aligned} \vec{a} &= a_x \hat{i} + a_y \hat{j} + a_z \hat{k} \\ \vec{b} &= b_x \hat{i} + b_y \hat{j} + b_z \hat{k} \end{aligned}$$

  - `c=cross(a,b);`
  - Rules:
    - If ordering is *forward* (*i j k i j k*) then sign is *positive*.
    - If ordering is *backward* (*k j i k j i*) then sign is *negative*.
    - *ii jj kk* are *zero*.
  - The "right-hand rule"





A few other useful tidbits:

$$\left|\vec{a} \times \vec{b}\right| = |\vec{a}||\vec{b}|\sin\theta$$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$