The Desk Reference of Statistical Quality Methods

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Designed Experiments

Designed experiments provide a statistical tool to allow for the efficient examination of several input *factors* and to determine their effect on one or more *response* variables.

Input factors are those characteristics or operating conditions over which we may or may not have direct control but that can and do affect a process. Response variables or factors are the observations or measurements that result from a process.

Examples of this input factor/response variable relationship follow.

Process: Wood gluing operation

Input factors	*Response variables*
Amount of glue	Tensile strength, pounds/inch^2
Type of glue	
Drying temperature	
Drying time	
Moisture content of wood	
Type of wood	
Relative humidity of environment	

For all of these input factors, there is an optimum level of setting that will maximize the bond strength. Some of these factors may be of more importance than others.

Designed experiments (frequently referred to as design of experiments, or DOE) can assist in determining which factors play a role in affecting the level of response.

There are essentially the following five steps in a DOE:

1. Brainstorming to identify potential input factors (or factors) and output responses (or responses) and establishing *levels* for the factors and the measure for the response(s)
2. Designing the experimental design or *matrix*
3. Performing the experiment
4. Analyzing the results
5. Performing a validation run to test the results

Example A:
A manufacturer of plastic/paper laminate wants to investigate the lamination process to see if improvements to the process can be made.

Step 1. Brainstorm.

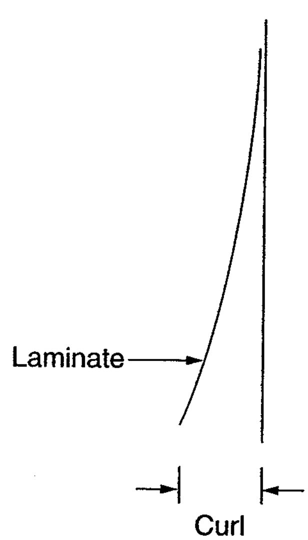
During this initial stage, individuals gather to discuss and define input factors and output responses. The results of this meeting yield the following:

Input factors	*Response variables*
Top roll tension setting, lb.	Amount of curl
Rewind tension setting, lb.	Number of wrinkles per 100 ft.
Bottom roll tension setting, lb.	Peel strength, lb./in.
Take-up speed, ft./min.	
Type of paper	
Thickness of plastic film, mils	
Type of plastic film	

For this initial experiment, three factors and one response variable are selected. The input factors are identified as *A, B,* and *C*. The response variable is the amount of curl, and the three input factors are

Input factor	*Description*
A	Top roll tension
B	Bottom roll tension
C	Rewind tension

Response variable	*Description*
Curl, inches	A 36-inch strip of paper/plastic film is hung against a vertical plane. The distance from the end of the laminate to the vertical plane is measured in millimeters.



The objective of this experiment is to better understand the effects of the three chosen factors on the amount of curl in the laminate. The current method calls for these factors to be set as follows:

Factor	*Setting*
A	22
B	22
C	9

For this experiment, we will set each of the factors to a "+" level and a "–" level. The notation for this three-factor, two-level experiment is

$$2^3$$

where: 2 = number of levels
3 = number of factors

The total number of experimental conditions is given by one $2^3 = 8$.

The general case for any two-level experiment, where all possible experimental conditions are run is given by

$$2^n$$

where: n = the number of factors

Each of the factors will be set at a + and – level around the traditional value at which they are run.

Note: Some text on DOE will use *1* for one level and *2* for the second level.

Factor	*– Setting*	*+ Setting*
A	16	28
B	16	28
C	6	12

The new conditions are set at such a level as to give a change in the response variable if the factor is a contributor to the response variable. The change in the factor setting should be large enough to change the response variable but not enough to critically affect the process.

Step 2. Design the experiment.

This experiment is a full 2^3 and will have eight experimental setups. Start by having eight *runs* and three columns for the factors *A, B,* and *C*.

	Factors		
Run	***A***	***B***	***C***
1			
2			
3			
4			
5			
6			
7			
8			

Start by placing a "–" for the first run of factor *A* and then alternating + and – for the first column.

	Factors		
Run	***A***	***B***	***C***
1	–		
2	+		
3	–		
4	+		
5	–		
6	+		
7	–		
8	+		

The second column is treated in a similar manner, except there are pairs of –'s and +'s. The third column is completed with groups of four –'s and four +'s. This completed table represents the design or matrix for the two-level, three-factor experiment, or the 2^3 design.

	Factors		
Run	***A***	***B***	***C***
1	–	–	–
2	+	–	–
3	–	+	–
4	+	+	–
5	–	–	+
6	+	–	+
7	–	+	+
8	+	+	+

The order of one through eight listed in numerical sequence is called the standard order. The actual sequence for the eight experiments should be a random order. The – and + signs determine the setting for the particular run.

For example, run #3 would have:

Factor *A* (Top roll tension) set at 16

Factor *B* (Bottom roll tension) set at 28

Factor *C* (Rewind tension) set at 6

Step 3. Perform the experiment.

Each of the eight experimental runs is made in a random order. The results of each run are recorded, and a single measurement observation will be made for each run.

Step 4. Analyze the data.

Main Effects:

For each of the main factors *A, B,* and *C,* an *effect* will be determined. The effect will be the average effect obtained when the factor under consideration is changed from a – setting to a + setting. The signs of the column to which a factor has been assigned will be used to determine the effect.

Step 5. Validation.

The original objective was to minimize the curl response. From the graphical analysis of the effects we have concluded that factors *A* and *C* are major factors with respect to their effect on curl. Since we are minimizing the response of curl, we must set the factors *A* and *C* to the + and – settings respectively. The top roll tension should be set to 28, the re-wind tension set to 6, and the remaining factor *B* to the most economical setting as it has no significant effect on tension.

The process should now be run for a more extended period of time to collect data on the curl response. This data from the improved process can now be compared to the historical data using traditional methods of hypothesis testing. Essentially we want to know if there has been a measured, statistically significant reduction in the amount of curl.

	Factors			
Run	***A***	***B***	***C***	**Response, curl**
1	–	–	–	87
2	+	–	–	76
3	–	+	–	90
4	+	+	–	83
5	–	–	+	101
6	+	–	+	92
7	–	+	+	100
8	+	+	+	92

Effects are determined by taking the difference between the average response when the factor is set + and the average response when the factor is set –.

Main Effect *A*:

$$\left(\frac{76+83+92+92}{4}\right)-\left(\frac{87+90+101+100}{4}\right)=85.75-94.50=-8.75$$

The average effect in going from a + setting for factor *A* to a – setting for factor *A* is –8.75 units of curl.

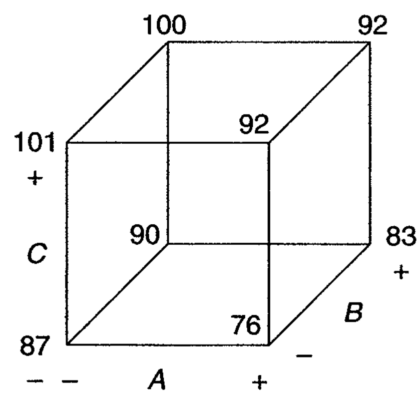
Main Effect *B*:

$$\left(\frac{90+83+100+92}{4}\right)-\left(\frac{87+76+101+92}{4}\right)=91.25-89.00=+2.25$$

Main Effect *C*:

$$\left(\frac{101+92+100+92}{4}\right)-\left(\frac{87+76+90+83}{4}\right)=96.25-84.00=+12.25$$

The main effects can be visualized by drawing a cube plot of the responses.



It can be seen that in order to minimize the curl response, the following setting for the factors should be made:

Factor	*Setting*
Factor *A* (Top roll tension)	$A+=28$
Factor *B* (Bottom roll tension)	$B-=16$
Factor *C* (Rewind tension)	$C-=6$

In addition to the main effects, there can be interaction effects. Interactions result when the effect of one main effect depends on or is related to the effect of another effect. For example, the rate at which a chemical reaction occurs can be influenced by a catalyst. The selection of the catalyst can be a determining factor. Catalyst *A* might perform well but only at a higher temperature, where catalyst *B* would perform as well as catalyst *A* but only at a lower temperature. The catalyst and temperature factors interact with each other; in other words, there is a catalyst-temperature interaction.

In order to evaluate interactions of the three factors *A, B,* and *C,* the interactions will be determined. There are three interactions that involve two factors:

AB = the interaction of top roll tension and bottom roll tension

BC = the interaction of bottom roll tension and rewind tension

AC = the interaction of top roll tension and rewind tension

These interactions may also be written as $A \times B$, $B \times C$, and $A \times C$. In addition, there is a single three-factor interaction expressed as *ABC.*

The computation of these interactions is performed using the signs of the columns as was the case with the main effects. The signs for the interaction columns are determined by multiplying the signs of the main effects used in the interaction.

	Factors							
Run	***A***	***B***	***C***	***AB***	***AC***	***BC***	***ABC***	**Response, curl**
1	−	−	−	+	+	+	−	87
2	+	−	−	−	−	+	+	76
3	−	+	−	−	+	−	+	90
4	+	+	−	+	−	−	−	83
5	−	−	+	+	−	−	+	101
6	+	−	+	−	+	−	−	92
7	−	+	+	−	−	+	−	100
8	+	+	+	+	+	+	+	92

***AB* effect:**

$$\left(\frac{87+83+101+92}{4}\right)-\left(\frac{76+90+92+100}{4}\right)=90.75-89.50=+1.25$$

***AC* effect:**

$$\left(\frac{87+90+92+92}{4}\right)-\left(\frac{76+83+101+100}{4}\right)=90.25-90.00=+0.25$$

***BC* effect:**

$$\left(\frac{87+76+100+92}{4}\right)-\left(\frac{90+83+101+92}{4}\right)=88.75-91.50=-2.75$$

***ABC* effect:**

$$\left(\frac{76+90+101+92}{4}\right)-\left(\frac{87+83+92+100}{4}\right)=89.75-90.50=-0.75$$

Are these effects significant, or are the differences due to random, experimental error?

There are two ways to determine if these effects are statistically significant. One way is to repeat the experiment and determine the actual experimental error, and the other is utilized when only one observation is made for each run (as is this case). When only one observation per run is available, the use of a normal probability plot can be helpful.

NORMAL PROBABILITY PLOT

Normal probability plotting paper is used. The effect is located on the horizontal axis, and the relative rank of the effect is expressed as a percent median rank (%MR).

Step 1. Arrange all of the effects in ascending order (starting with the lowest at the top). The order of this ranking is expressed as *i*.

Order *i*	Effect value	Effect
1	–8.75	*A*
2	–2.75	*BC*
3	–0.75	*ABC*
4	0.25	*AC*
5	1.25	*AB*
6	2.25	*B*
7	12.25	*C*

Note that there is one less effect than there are experimental runs.

Step 2. Calculate the percent median rank values.

$$\%\text{MR} = \frac{i-.3}{n+.4} \times 100$$

where: i = the order of the effect
n = the total number of effects

This median rank is called the Benard Median Rank and follows the general expression of $\text{MR} = \dfrac{i-c}{n-2c+1}$ where $c = 0.3$. Frequently another form of the median rank is the Hazen, where $c = 0.5$.

$$\text{First \% median rank} = \%\text{MR} = \frac{i-.3}{n+.4} \times 100 = \%\text{MR} = \frac{1-.3}{7+.4} \times 100 = \frac{0.7}{7.4} \times 100 = 9.5$$

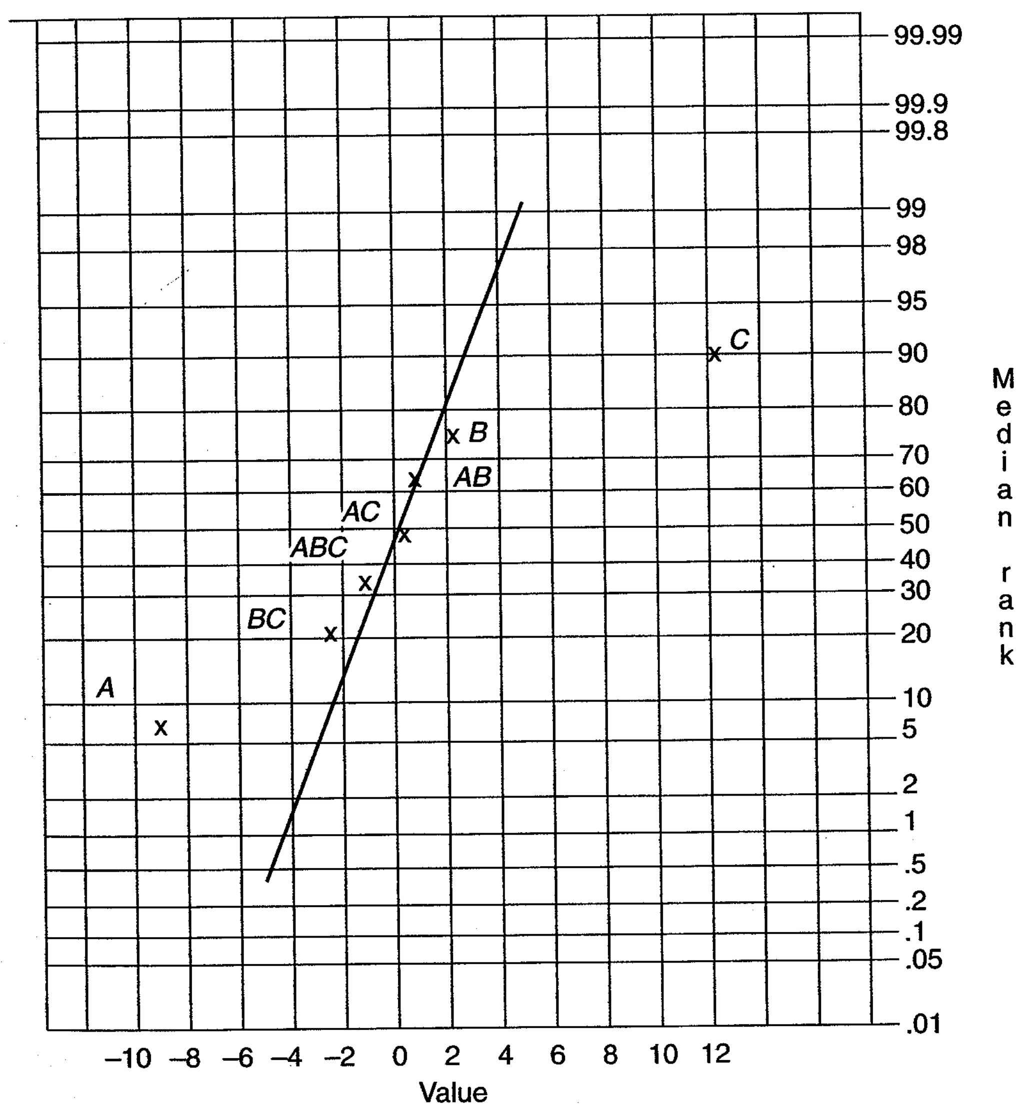
$$\text{Second \% median rank} = \frac{2-.3}{7+.4} \times 100 = 23.0$$

$$\text{Third \% median rank} = \frac{3-.3}{7+.4} \times 100 = 36.5$$

Order *i*	Effect value	Effect	Percent median rank
1	–8.75	*A*	9.5
2	–2.75	*BC*	23.0
3	–0.75	*ABC*	36.5
4	0.25	*AC*	50.0
5	1.25	*AB*	63.5
6	2.25	*B*	77.0
7	12.25	*C*	90.5

Step 3. Plot the effects as a function of the percent median rank on the normal probability paper.

Draw a straight line through the points, concentrating the emphasis around the points nearest the zero effect. Effects falling off of this straight line are judged to be significant. To maximize a response variable, set the main effect to the same sign as the effect. To minimize the response variable, set the factor to the reverse of the effect. In this example, factors *A* and *C* are statistically significant.



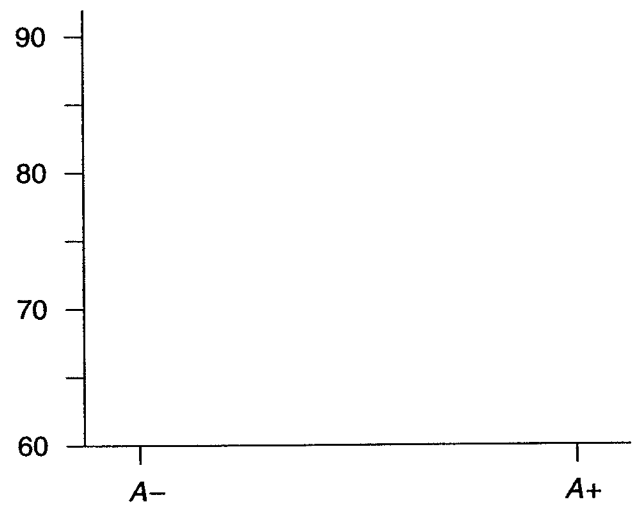
VISUALIZATION OF TWO-FACTOR INTERACTIONS

Main effects can be visualized using the cube plot. Two-factor interactions can be visualized using linear plots.

The following demonstrates how to develop the two-factor interaction for the *AB* interaction.

Step 1. Draw a two-axis graph.

The response will be plotted on the vertical axis, and one of the two main effects involved in the two-factor interaction will be plotted on the horizontal axis. The horizontal value is attribute in nature. Only the designations of – and + are used for the selected factor.



Since factor *A* was chosen for the horizontal axis, the remaining factor *B* will be plotted as a pair of linear lines—one for *B*– and one for *B*+.

The *B*– line will be plotted first. There are two experimental runs where *A* is set to *A*– and factor *B* is set to *B*–. These two runs are run #1 and run #5. The average response for these two is:

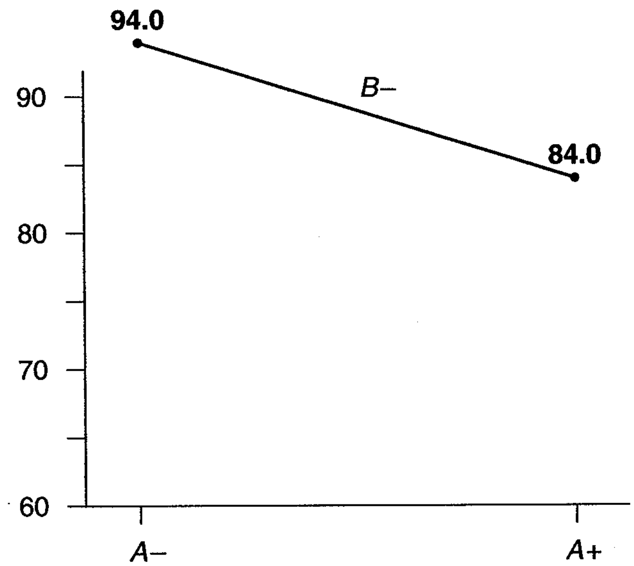
$$\frac{87+101}{2}=94.0$$

When factor *A* is set to *A*– and factor *B* is set to *B*–, the average response is 94.0.

The other end of the *B*– line is determined by calculating the average response when *A* is set at *A*+ and *B* is set at *B*–. *A*+ and *B*– are found in two experimental runs—run #2 and run #6. The average response for these two is

$$\frac{76+92}{2}=84.0$$

These two values define the *B*– line on the graph.

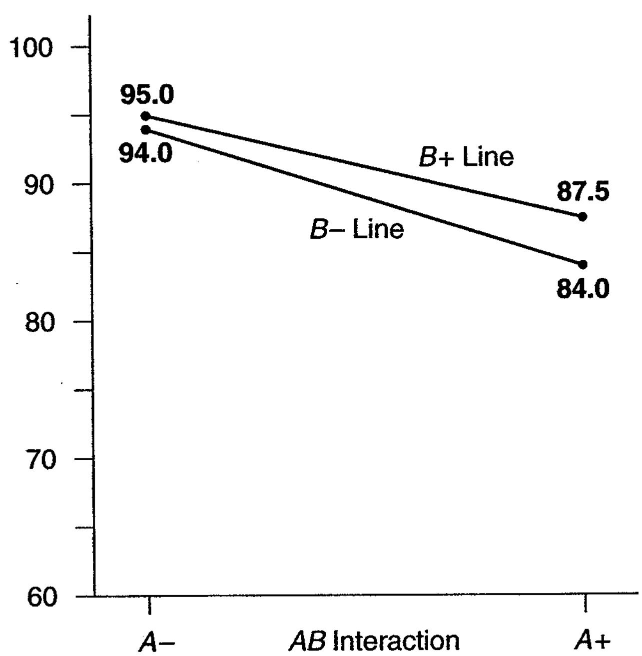


:he remaining line is the $B+$ line. There are two runs where B is set + and A is set –. e are run #3 and run #7. The average of these two responses is

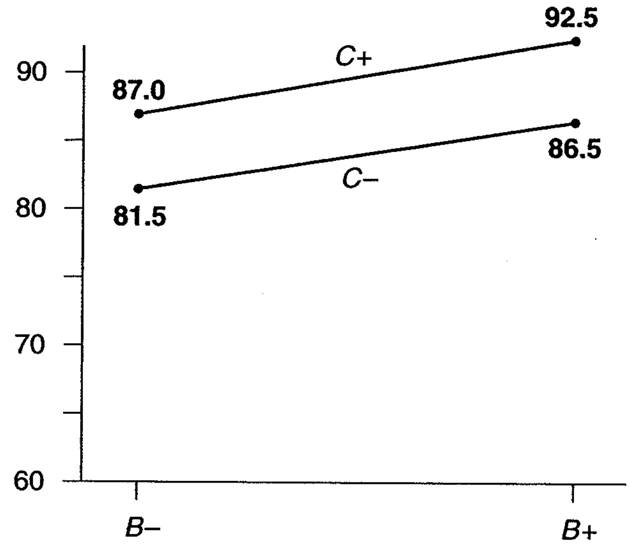
$$\frac{90+100}{2}=95.0$$

[he other end of the $B+$ line is at the $A+$ position. There are two runs where B is set + 1 is set +. These are run #4 and run #8. The average of these two responses is:

$$\frac{83.0+92.0}{2}=87.5$$



Plotting the *B*+ line completes the two-factor interaction plot.



The linear graph for the *BC* interaction follows:

The C– line:

$$C-B-\text{point}=\frac{87.0+76.0}{2}=81.5$$

$$C-B+\text{point}=\frac{90.0+83.0}{2}=86.5$$

The C+ line:

$$C+B-\text{point}=\frac{93.0+81.0}{2}=87.0$$

$$C+B+\text{point}=\frac{95.0+90.0}{2}=92.5$$

In the original experiment, only one observation was made for each experiment run. The only technique to determine if the effects were statistically significant was to perform a normal probability plot and make a subjective judgment if any of the points fell off of the straight line (with emphasis on the zero point). This method is subject to errors of judgment, but it is the best technique available when only one observation is available.

In order to determine experimental error, the entire experiment must be replicated at least one more time in order to calculate a measure of variation—the standard deviation.

The entire set of eight runs is replicated. The new data from the first and second replications yield an average and standard deviation for each of the eight runs:

	Factors										
Run	*A*	*B*	*C*	*AB*	*AC*	*BC*	*ABC*	1st	2nd	Avg.	Std. dev.
1	–	–	–	+	+	+	–	87	88	87.5	0.707
2	+	–	–	–	–	+	+	76	78	77.0	1.414
3	–	+	–	–	+	–	+	90	92	91.0	1.414
4	+	+	–	+	–	–	–	83	80	81.5	2.121
5	–	–	+	+	–	–	+	101	96	98.5	3.535
6	+	–	+	–	+	–	–	92	91	91.5	0.707
7	–	+	+	–	–	+	–	100	104	102.0	2.828
8	+	+	+	+	+	+	+	92	91	91.5	0.707

The new calculated effects, in decreasing order, based on the average of two observations/run are:

Factor	Effect
A	–9.38
ABC	–1.13
BC	–1.13
AB	–0.63
AC	0.63
B	2.88
C	11.63

While a normal probability plot would yield the same conclusion that factor *A* and factor *B* are probably statistically significant, again, the conclusion is somewhat subjective.

By using the *t*-distribution and determining the confidence interval for each of the effects, a better judgment can be made.

The following method can test if the effect is within the normal variation of the experimental error or if the effect is statistically removed from the limits of normal variation. That is, the effect is real and not just a random value.

A confidence interval will be determined for each effect. If zero is found to be within the confidence interval limits, the conclusion will be that the observed effect is simply random variation within the expectation for the calculated experimental error. If, however, the confidence interval does not contain zero, the conclusion will be made that the effect is real and is statistically significant.

Calculate the pooled standard deviation for all the observations.

$$Sp = \sqrt{\frac{V_1 S_1^2 + V_2 S_2^2 + \ldots\ V_n S_n^2}{V_1 + V_2 + \ldots\ V_n}}$$

where: V_1 = number of observations in run #1
$S^2{}_1$ = the variance (standard deviation squared) for sample run #1
V_n = number of observations in the nth run
$S^2{}_n$ = variance of the nth sample run

$$Sp = \sqrt{\frac{(2)(0.707)^2 + (2)(1.414)^2 + (2)(1.414)^2 + \ldots + (2)(2.828)^2 + (2)(0.707)^2}{2+2+2+2+2+2+2+2}}$$

$$Sp = \sqrt{\frac{0.9997 + 3.9988 + 3.9988 + 8.9973 + 24.9925 + 0.9997 + 15.9952 + 0.9997}{16}}$$

$$Sp = 1.952$$

Note: If the number of replicates is the same for all runs, then the pooled standard deviation is simply the square root of the average variances.

Each of the calculated effects is actually a point estimate. The confidence interval for each of these estimated effects is determined by

$$\text{Effect} \pm \text{Error}$$

The error is calculated as:

$$t_{\alpha/2,(r-1)2^{k-f}} \left(Sp\sqrt{\frac{2}{pr}} \right)$$

where: α = risk (Confidence = 1 – Risk)
p = number of +'s per effect column
r = number of replicates
Sp = pooled standard deviation
f = degree of fractionation

The error of the estimated effect at a level of confidence of 95 percent is

$$E = t_{\alpha/2,(r-1)2^{k-f}} \left(Sp\sqrt{\frac{2}{pr}} \right)$$

$$t_{\alpha/2,(r-1)2^{k-f}} = t_{0.025,8} = 2.306$$

$$E = 2.306\left(1.952\sqrt{\frac{2}{(4)(2)}} \right) \qquad E = 2.25$$

Any effect that is contained within the limits of 0 ± 2.25 is considered to be not statistically significant.

The following effects are significant:

Effect	*95 percent confidence interval*
A	–11.63 to –7.13
B	0.63 to 5.13
C	8.75 to 13.88

ote: Any effect whose absolute value is greater than the error is statistically significant.

The main effect *B* is deemed to be statistically significant at the 95 percent level of nfidence. This was not evident using the normal probability plot method.

FRACTIONAL FACTORIAL EXPERIMENTS

four factors are to be investigated at two levels each, then the total number of experi-ents required would be $2^4 = 16$, and in order to establish an experimental error, two repli-tes will be required for a total of 32 observations. This number of observations and periments can be time consuming and cost prohibitive in many cases.

The same number of factors can be evaluated using half the normal runs by running a e-half fractional factorial experiment.

Consider the requirement for a full four-factor experiment. This design can be created extending the eight runs to 16 and adding a fourth column for the next factor *D*.

	Factors			
Run	***A***	***B***	***C***	***D***
1	–	–	–	–
2	+	–	–	–
3	–	+	–	–
4	+	+	–	–
5	–	–	+	–
6	+	–	+	–
7	–	+	+	–
8	+	+	+	–
9	–	–	–	+
10	+	–	–	+
11	–	+	–	+
12	+	+	–	+
13	–	–	+	+
14	+	–	+	+
15	–	+	+	+
16	+	+	+	+

Three-level interactions are very rare. If the assumption is made that any three-level interaction is actually due to experimental error, then by letting the *ABC* interaction of this 2^4 experiment equal the *D* main effect, the full 16 runs for a full 2^4 can be cut in half. In other words, let $D = ABC$. $D = ABC$ is called a *generator*, and the *identity* is $I = ABCD$.

By mixing the effect of main effect *D* with the three-level interaction *ABC*, we have confounded the two effects. *D* and *ABC* are called *aliases*. Since *D* and *ABC* are aliases, we cannot differentiate between *D* and *ABC*.

	Factors				
Run	***A***	***B***	***C***	***D***	***ABC***
1	−	−	−	−	−
2	+	−	−	−	+
3	−	+	−	−	+
4	+	+	−	−	−
5	−	−	+	−	+
6	+	−	+	−	−
7	−	+	+	−	−
8	+	+	+	−	+
9	−	−	−	+	−
10	+	−	−	+	+
11	−	+	−	+	+
12	+	+	−	+	−
13	−	−	+	+	+
14	+	−	+	+	−
15	−	+	+	+	−
16	+	+	+	+	+

Let $D = ABC$.

	Factors			***D***
Run	***A***	***B***	***C***	***ABC***
1	−	−	−	−
2	+	−	−	+
3	−	+	−	+
4	+	+	−	−
5	−	−	+	+
6	+	−	+	−
7	−	+	+	−
8	+	+	+	+
9	−	−	−	−
10	+	−	−	+
11	−	+	−	+
12	+	+	−	−
13	−	−	+	+
14	+	−	+	−
15	−	+	+	−
16	+	+	+	+

Notice that run #1 through run #8 are now duplicated by run #9 through run #16. Run #9 through run #16 can be eliminated. Four factors can be run using eight experimental runs. This experimental design is called a 2^{4-1} design.

Fractional factorial designs can be designated by the general form of

$$2^{k-f}$$

where: k = number of factors 2^{k-f}
f = degree of fractionation
f = 1 or ½ fractional design
f = 2 or ¼ fractional design
f = 3 or ⅛ fractional design

The number of experimental runs is determined by the value of 2^{k-f}. In a 2^{7-3}, there are 16 runs—$2^4 = 16$.

The *resolution* of a design determines the degree to which confounding is present. Resolutions are expressed as roman numerals III, IV, or V and are named by the number of letters in the shortest word of the defining relationship. In the example of a 2^{4-1}, the defining relationship was $D = ABC$. The identity element is $I = ABCD$. There are four letters in the word; therefore, the resolution is IV.

The proper designation for this design is:

$$2_{IV}^{4-1}$$

Possible experimental design resolutions are as follows:

R_{III}: Main effects and two-factor interactions are confounded, so be careful.

R_{IV}: Main effects are clear from any two-factor interactions, ~~and~~ but two-factor interactions are confounded. with other two-Factor Interactions

R_{V}: Main effects are clear, and two-factor interactions are clear.

Example:
In a 2^{5-2}, the defining relationships are: $D = AB$ and $E = AC$.

This is a 1/4 fractional design. The number of factors is five, the number of runs is eight, and the resolution is III ($I = ABD$ and $I = ACE$), three letters make up the identity element.

This is a 2_{III}^{5-2} experiment.

Table 1. Selected Fractional Factorial Designs.

Number of factors *k*	Designation	Number of runs	Design generator
3	2_{III}^{3-1}	4	$A = BC$
4	2_{IV}^{4-1}	8	$D = ABC$
5	2_{V}^{5-1}	16	$E = ABCD$
	2_{III}^{5-2}	8	$D = AB$
			$E = AC$
6	2_{VI}^{6-1}	32	$F = ABCDE$
	2_{IV}^{6-2}	16	$E = ABC$
			$F = BCD$
7	2_{VII}^{7-1}	64	$G = ABCDEF$
	~~2_{IV}^{7-2}~~ 2_{V}^{7-2}	32	$F = ABCD$
			$G = ABDE$
	2_{IV}^{7-3}	16	$E = ABC$
			$F = BCD$
			$G = ACD$
	2_{III}^{7-4}	8	$D = AB$
			$E = AC$
			$F = BC$
			$G = ABC$

Case study:

The AstroSol company manufactures solar panels used to generate electricity. The manufacturing process consists of deposition of alternating layers of tin (Sn) and cadmium telluride (CdTe) and then baking these deposited layers in an oven. The power output is then measured and reported in units of watts/ft^2.

The R&D department wants to maximize the output of the cells. After considerable review and discussion, the design team decides to look at five factors in a 1/2 fractional factorial design, 2_V^{5-1}, as this design will isolate a main and two factorial interactions.

The five factors to be considered are:

Thickness of Sn in microns

Thickness of CdTe in microns

Bake temperature, °F

Baking time, minutes

Source of CdTe

All other factors in the process will remain fixed at their current levels.

A relative – and + for each of the factors around the current operating conditions is determined for each factor.

Factor	Description	–	+
A	Thickness of Sn, μ	20	40
B	Thickness of CdTe, μ	30	50
C	Bake temperature, °F	350	400
D	Bake time, minutes	40	50
E	Source of CdTe	United States	China

For a 2_V^{5-1} fractional design, the base design will be that of a 2^4 full factorial where the fifth factor will be generated by $E = ABCD$ and $I = ABCDE$.

Interactions:

$AB = CDE$ $\quad AE = BCD$ $\quad BE = ACD$ $\quad DE = ABC$

$AC = BDE$ $\quad BC = ADE$ $\quad CD = ABE$

$AD = BCE$ $\quad BD = ACE$ $\quad CE = ABD$

Since this is a resolution V, all main effects are clear and all two-factor interactions are clear.

The 16 runs were replicated three times so that experimental error and statistical significance can be determined. The individual responses, response average, and standard deviation are recorded in Table 2.

Calculation of Main Effects

***A* effect:**

$$\left(\frac{9.23+9.76+8.27+9.83+8.23+10.17+8.30+9.50}{8}\right)$$
$$-\left(\frac{9.30+10.73+9.30+9.73+10.10+10.97+8.30+10.77}{8}\right)=-0.74$$

***B* effect:**

$$\left(\frac{10.73+9.76+9.73+9.83+10.97+10.17+10.77+9.50}{8}\right)$$
$$-\left(\frac{9.30+9.23+9.30+8.27+10.10+8.23+8.50+8.30}{8}\right)=1.28$$

***C* effect:**

$$\left(\frac{9.30+8.27+9.73+9.83+8.50+8.30+10.77+9.50}{8}\right)$$
$$-\left(\frac{9.30+9.23+10.73+9.76+10.10+8.23+10.97+10.17}{8}\right)=-0.54$$

Table 2. Design Matrix.

Run	A	B	C	D	$ABCD$ E	CDE AB	BDE AC	BCE AD	BCD AE	ADE BC	ACE BD	ACD BE	ABE CD	ABD CE	ABC DE	Response Y_1	Y_2	Y_3	Avg.	S
1	−	−	−	−	+	+	+	+	−	+	+	−	+	−	−	9.3	9.4	9.2	9.30	0.10
2	+	−	−	−	−	−	−	−	−	+	+	+	+	+	+	9.1	9.4	9.2	9.23	0.15
3	−	+	−	−	−	−	+	+	+	−	−	−	+	+	+	10.8	10.8	10.6	10.73	0.12
4	+	+	−	−	+	+	−	−	+	−	−	+	+	−	−	9.8	9.6	9.9	9.76	0.15
5	−	−	+	−	−	+	−	+	+	−	+	+	−	−	+	9.5	9.0	9.4	9.30	0.26
6	+	−	+	−	+	−	+	−	+	−	+	−	−	+	−	8.0	8.4	8.4	8.27	0.23
7	−	+	+	−	+	−	−	+	−	+	−	+	−	+	−	9.9	9.6	9.7	9.73	0.15
8	+	+	+	−	−	+	+	−	−	+	−	−	−	−	+	9.8	9.9	9.8	9.83	0.06
9	−	−	−	+	−	+	+	−	+	+	−	+	−	+	−	10.4	10.5	9.4	10.10	0.61
10	+	−	−	+	+	−	−	+	+	+	−	−	−	−	+	8.5	8.0	8.2	8.23	0.25
11	−	+	−	+	+	−	+	−	−	−	+	+	−	−	+	10.8	11.1	11.0	10.97	0.15
12	+	+	−	+	−	+	−	+	−	−	+	−	−	+	−	10.2	9.7	10.6	10.17	0.45
13	−	−	+	+	+	+	−	−	−	−	−	−	+	+	+	8.6	8.2	8.7	8.50	0.26
14	+	−	+	+	−	−	+	+	−	−	−	+	+	−	−	7.8	8.4	8.7	8.30	0.46
15	−	+	+	+	−	−	−	−	+	+	+	−	+	−	−	11.0	10.6	10.7	10.77	0.21
16	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	9.2	9.9	9.4	9.50	0.36

D **effect:**

$$\left(\frac{10.10+8.23+10.97+10.17+8.50+8.30+10.77+9.50}{8}\right)$$
$$-\left(\frac{9.30+9.23+10.73+9.76+9.30+8.27+9.73+9.83}{8}\right)=0.05$$

E **effect:**

$$\left(\frac{9.30+9.76+8.27+9.73+8.23+10.97+8.50+9.50}{8}\right)$$
$$-\left(\frac{9.23+10.73+9.30+9.83+10.10+10.17+8.30+10.77}{8}\right)=-0.52$$

Two-Factor Interactions

AB **interaction:**

$$\left(\frac{9.30+9.76+9.30+9.83+10.10+10.17+8.50+9.50}{8}\right)$$
$$-\left(\frac{9.23+10.73+8.27+9.73+8.23+10.97+8.30+10.77}{8}\right)=0.07$$

AC **interaction:**

$$\left(\frac{9.30+10.73+8.27+9.83+10.10+10.97+8.30+9.50}{8}\right)$$
$$-\left(\frac{9.23+9.76+9.30+9.73+8.23+10.17+8.50+10.77}{8}\right)=0.16$$

AD **interaction:**

$$\left(\frac{9.30+10.73+9.30+9.73+8.23+10.17+8.30+9.50}{8}\right)$$
$$-\left(\frac{9.23+9.76+8.27+9.83+10.10+10.97+8.50+10.77}{8}\right)=-0.27$$

AE **interaction:**

$$\left(\frac{10.73+9.76+9.30+8.27+10.10+8.23+10.77+9.50}{8}\right)$$
$$-\left(\frac{9.30+9.23+9.73+9.83+10.97+10.17+8.50+8.30}{8}\right)=0.08$$

***BC* interaction:**

$$\left(\frac{9.30+9.23+9.73+9.83+10.10+8.23+10.77+9.50}{8}\right)$$
$$-\left(\frac{10.73+9.76+9.30+8.27+10.97+10.17+8.50+8.30}{8}\right)=0.11$$

***BD* interaction:**

$$\left(\frac{9.30+9.23+9.30+8.27+10.97+10.17+10.77+9.50}{8}\right)$$
$$-\left(\frac{10.73+9.76+9.73+9.83+10.10+8.23+8.50+8.30}{8}\right)=0.29$$

***BE* interaction:**

$$\left(\frac{9.23+9.76+9.30+9.73+10.10+10.97+8.30+9.50}{8}\right)$$
$$-\left(\frac{9.30+10.73+8.27+9.83+8.23+10.17+8.50+10.77}{8}\right)=0.14$$

***CD* interaction:**

$$\left(\frac{9.30+9.23+10.73+9.76+8.50+8.30+10.77+9.50}{8}\right)$$
$$-\left(\frac{9.30+8.27+9.73+9.83+10.10+8.23+10.97+10.17}{8}\right)=-0.06$$

***CE* interaction:**

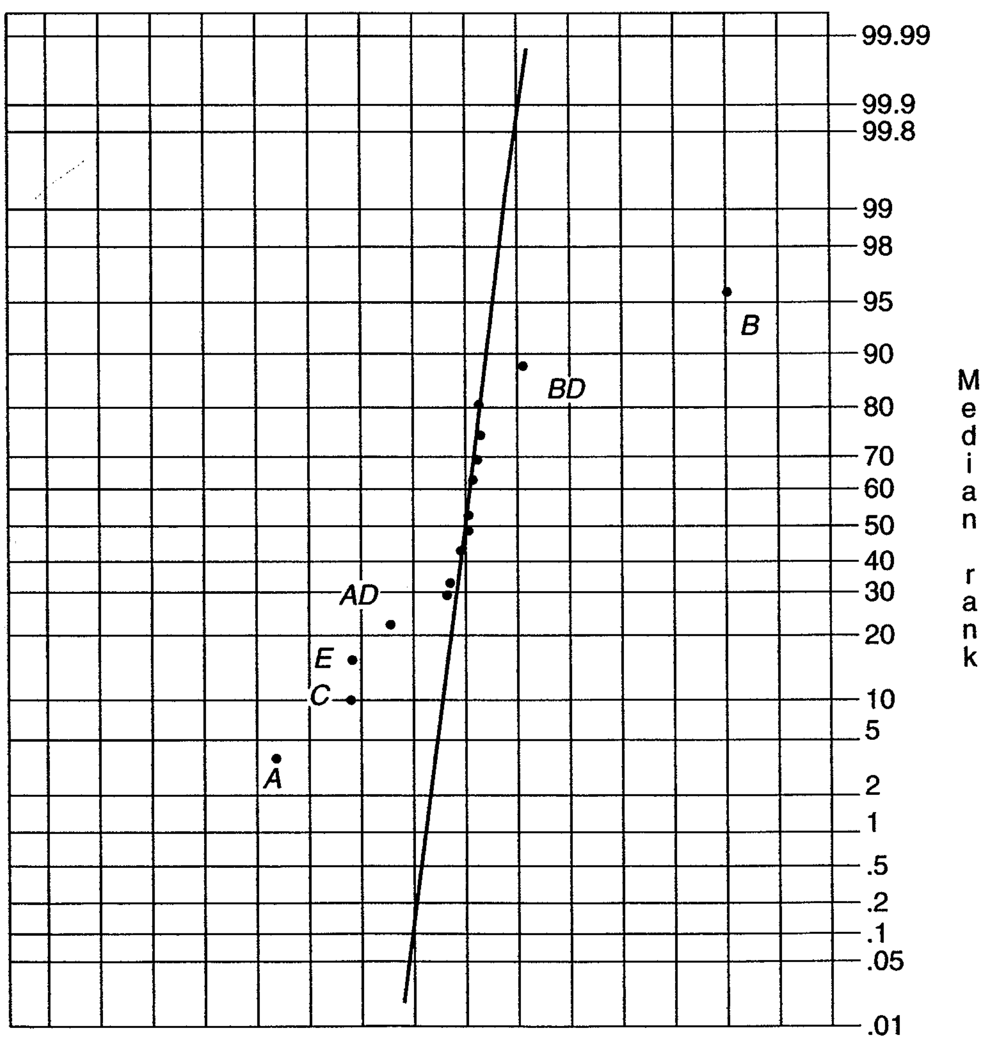
$$\left(\frac{9.23+10.73+8.27+9.73+10.10+10.17+8.50+9.50}{8}\right)$$
$$-\left(\frac{9.30+9.76+9.30+9.83+8.23+10.97+8.30+10.77}{8}\right)=-0.03$$

***DE* interaction:**

$$\left(\frac{9.23+10.73+9.30+9.83+8.23+10.97+8.50+9.50}{8}\right)$$
$$-\left(\frac{9.30+9.76+8.27+9.73+10.10+10.17+8.30+10.77}{8}\right)=-0.01$$

A summary of all main and two-factor interaction effects are ranked in ascending order and percent median rank values are calculated as follows:

Order *i*	Effect value	Effect	Percent median rank
1	–0.74	*A*	4.5
2	–0.54	*C*	11.0
3	–0.52	*E*	17.5
4	–0.27	*AD*	24.0
5	–0.06	*CD*	30.5
6	–0.03	*CE*	37.0
7	–0.01	*DE*	43.5
8	0.05	*D*	50.0
9	0.07	*AB*	56.5
10	0.08	*AE*	63.0
11	0.11	*BC*	69.5
12	0.14	*BE*	76.0
13	0.16	*AC*	82.5
14	0.29	*BD*	89.0
15	1.28	*B*	95.5



The effects are plotted on normal probability paper to estimate which effects and interactions are significant.

A normal probability plot of effects is shown in the following figure:

Based on the analysis of the normal probability plot, it appears that main effects *A, C, E,* and *B* are statistically significant. In addition, consideration should be given to the *BD* and *AD* interactions.

In order to maximize the power output factors, *A, C,* and *E* should be set at the – level and factor *B* at the + level.

Statistical Significance Based on *t* test

Step 1. Calculate the pooled standard deviation.

$$Sp = \sqrt{\frac{V_1S_1^2 + V_2S_2^2 + \ldots\ldots\ldots V_nS_n^2}{V_1 + V_2 + \ldots\ldots\ldots V_n}}$$

$$Sp = \sqrt{\frac{(3)(0.10)^2 + (3)(0.15)^2 + (3)(0.12)^2 + \ldots\ldots\ldots (3)(0.21)^2 + (3)(0.36)^2}{3+3+3+\ldots\ldots\ldots 3+3}}$$

$$Sp = \sqrt{\frac{3.9855}{48}} \quad Sp = 0.288$$

Step 2. Determine the statistical significance at the 90 percent level of confidence.

$$\text{Error, } E = t_{\alpha/2,(r-1)2^{k-f}}\left(Sp\sqrt{\frac{2}{pr}}\right)$$

$$\alpha = 0.10 \quad p = 8$$

$$f = 1 \quad r = 3 \quad k = 5$$

$$t_{\alpha/2,(r-1)2^{k-f}} = t_{.05,32} = 1.698$$

$$E = 1.698\left(0.288\sqrt{\frac{2}{(8)(3)}}\right) E = 0.14$$

Any effect ±0.14 that contains zero is judged not statistically significant at the 90 percent confidence level.

Order i	Effect value	Effect	90% confidence interval	Statistically significant?
1	–0.74	*A*	–0.88 to –0.60	Yes
2	–0.54	*C*	–0.68 to –0.40	Yes
3	–0.52	*E*	–0.66 to –0.38	Yes
4	–0.27	*AD*	–0.41 to –0.13	Yes
5	–0.06	*CD*	–0.20 to 0.08	No
6	–0.03	*CE*	–0.17 to 0.11	No
7	–0.01	*DE*	–0.15 to 0.13	No
8	0.05	*D*	–0.09 to 0.19	No
9	0.07	*AB*	–0.07 to 0.21	No
10	0.08	*AE*	–0.06 to 0.22	No
11	0.11	*BC*	–0.03 to 0.25	No
12	0.14	*BE*	0.00 to 0.28	No
13	0.16	*AC*	0.02 to 0.30	Yes
14	0.29	*BD*	0.15 to 0.43	Yes
15	1.28	*B*	1.14 to 1.42	Yes

Conclusion:

Factors *A, B, C,* and *E* appear to be statistically significant at the 90 percent confidence level. The following factor setting will maximize the output.

Set *A, C,* and *E* at the – level, and set factor *B* at the + level.

Factor	Description	Optimum setting
A	Thickness of Sn	20 microns
B	Thickness of CdTe	50 microns
C	Bake temperature	350°
E	Source of CdTe	United States
D	Bake time	Does not significantly affect power output; therefore, set a lower time of 40 minutes to reduce cycle time.

VARIATION REDUCTION

Variation reduction is accomplished when we reduce the standard deviation; however, standard deviations are not normally distributed and cannot be used directly as a response to minimize. One approach is to take multiple observations during an experimental run, calculate the standard deviation, and take the log, or ln. The resulting transformed response can then be treated as a normally distributed response and the effects calculated in the traditional manner. The objective of the experiment is to reduce the log *S*, or ln *S*. Dr. George Box (1978) of the University of Wisconsin, Madison, and his colleagues have published a series of papers suggesting the use of $-\log 10(s)$ as a response variable when the objective is to reduce variation.

An alternative is to simply calculate the variance (standard deviation squared) and compare the average variance for all of the – factor settings to the average of all of the

+ settings using an F test. If the calculated F value exceeds a critical F value, we conclude that the factor under consideration is statistically significant.

Example:
The objective is to minimize the variation of compressive strength of a cast concrete material. A full three-factor experiment (2^3) was run. The experiment was replicated three times, and the standard deviation was determined using the three observations for each of the eight runs.

Run	A	B	C	AB	AC	BC	ABC	Average	S	S^2
1	−	−	−	+	+	+	−	1900	3.9	15.21
2	+	−	−	−	−	+	+	1250	5.4	29.16
3	−	+	−	−	+	−	+	1875	15.1	228.01
4	+	+	−	+	−	−	−	1160	14.8	219.04
5	−	−	+	+	−	−	+	2100	10.3	106.09
6	+	−	+	−	+	−	−	1740	8.0	64.0
7	−	+	+	−	−	+	−	1940	13.6	184.96
8	+	+	+	+	+	+	+	1350	11.1	123.21

We now calculate an F value for each of the effects.

For effect *A*:

$$(\bar{S})^2_+ = \frac{29.16 + 219.04 + 64.0 + 123.21}{4} = 108.85$$

$$(\bar{S})^2_- = \frac{15.21 + 228.01 + 106.09 + 184.96}{4} = 133.57$$

$$F = \frac{(\bar{S})^2_{\text{larger}}}{(\bar{S})^2_{\text{smaller}}} \qquad F = \frac{133.57}{108.85} = 1.23$$

For effect *B*:

$$(\bar{S})^2_+ = \frac{228.01 + 219.04 + 184.96 + 123.21}{4} = 188.81$$

$$(\bar{S})^2_+ = \frac{15.21 + 29.16 + 106.09 + 64.0}{4} = 53.62$$

$$F = \frac{(\bar{S})^2_{\text{larger}}}{(\bar{S})^2_{\text{smaller}}} \qquad F = \frac{188.81}{53.62} = 3.52$$

The remaining F values are calculated in a similar manner.

Effect	F value	
A	1.23	
B	3.52	(Statistically significant at a confidence level of 90 percent)
C	1.03	
AB	1.09	
AC	1.25	
BC	1.75	
ABC	1.01	

A critical F value is determined. This value is looked up in an F table using one-half the degrees of freedom as calculated previously for the t value used in testing for statistical significance.

$$df = \frac{(r-1)2^{k-f}}{2}$$

$$df = 8$$

where: r = number of replicates = 3
k = number of factors = 3
f = degree of fractionation = 0 (this is a full factorial)

At a level of confidence of 0.90, the risk is 0.10; dividing the risk between the two levels, we have an $\alpha/2$ of 0.05.

The appropriate critical F value is found for $F_{.05,8,8} = 3.44$. Any effects that have a calculated F value greater than this are considered statistically significant. We can see that factor B is significant and that the average variance was smaller when factor B was set to the – level; therefore, we should set B at the – level to minimize variation. The other factors are not statistically significant contributors to variation.

In the previous example if the objective had been to maximize the compressive strength, the response would have been the average compressive strength. Calculation of all of these effects using the average compressive strength yields the following:

Factor	Effect
A	+579
B	–373
C	+236
AB	–74
AC	+104
BC	–109
ABC	–41

The calculated experimental error is determined to be

$$\text{Experimental error, } E = t_{\alpha/2,(r-1)2^{k-f}}\left(Sp\sqrt{\frac{2}{pr}}\right)$$

Sp = the average standard deviation

$$Sp = \sqrt{\frac{\Sigma S_i^{\,2}}{n}} = 10.38$$

Experimental error at 90 percent confidence = 7.4

$$t_{\alpha/2,(r-1)2^{k-f}}\left(Sp\sqrt{\frac{2}{pr}}\right) = t_{0.05,16}\left(10.38\sqrt{\frac{2}{(4)(3)}}\right) = (1.746)(10.38)(0.41) = 7.4$$

At the 90 percent level of confidence, all of the factors in their interactions are statistically significant. This is due to the fact that the magnitude of the effects is relatively large compared to the experimental error of ±7.4.

In this case to maximize the compressive strength, we would set factors *A* and *C* to the + level and factor *B* to the – level. Setting *B* to the – level also minimizes the variation.

RESPONSE MODELING

Having determined the factor effects, we can establish a model to predict responses.

The expected response is determined by taking the grand average of all responses and adding the contribution of the various effects.

$$\text{Expected response, } X = \text{Grand average} + \left(\frac{\text{Effect } A}{2}\right) + \left(\frac{\text{Effect } B}{2}\right) + \left(\frac{\text{Effect } C}{2}\right) + \ldots$$

Consider the following case. An experiment was run using a full-factorial, three-factor experiment. The effects were as follows:

Factor	Effect	The average of all responses was 125.
A	+60	
B	–44	
C	+30	
AB	–12	
AC	+8	
BC	+10	
ABC	–4	

What is the expected maximum given these effects?

In order to maximize the response, we would set the factors at *A*+, *B*–, and *C*+.

$$\text{Maximum expected value} = \text{Grand average} + \left(\frac{\text{Effect } A}{2}\right) + \left(\frac{\text{Effect } B}{2}\right) + \left(\frac{\text{Effect } C}{2}\right) + \ldots$$

$$\text{Maximum expected value} = 125 + \left(\frac{60}{2}\right) + \left(\frac{44}{2}\right) + \left(\frac{30}{2}\right) = 192$$

This model only features the main-effects contribution. If we include all of the interactions, we simply include their effects with each one multiplied by the appropriate signs for the settings of the main effects.

$$\text{Maximum expected value} = 125 + \left(\frac{60}{2}\right) + \left(\frac{44}{2}\right) + \left(\frac{30}{2}\right) + \left((A)(B)\frac{\text{Effect } AB}{2}\right)$$

$$+\left((B)(C)\frac{\text{Effect } BC}{2}\right) + \text{other terms}$$

where: (A) = setting for factor $A = (+1)$
(B) = setting for factor $B = (-1)$
(C) = setting for factor $C = (+1)$

The complete expression for the expected maximum value when the parameter settings are $A+$, $B-$, and $C+$ is

$$\text{Maximum expected} = 125 + 30 + 22 + 15 + \underset{(AB)}{(1)(-1)(-6)} + \underset{(AC)}{(1)(1)(4)} + \underset{(BC)}{(-1)(1)(5)} + \underset{(ABC)}{(1)(-1)(1)(2)}$$

$$\text{Maximum expected} = 125 + 30 + 22 + 15 + 6 + 4 - 5 - 2 = 195$$

Note: Only those effects that are statistically significant should be used in the model. For the previous example, it is assumed that all effects are significant.

The same approach is used when minimization is the objective. The signs for the effects are reversed in those cases. For example, if the main effects for the factors were $A-$, $B+$, and $C-$, then setting A to the + level, B to the – level, and C to the + level would minimize the response.

We may use this modeling approach to determine where to set parameters in order to achieve a target value.

Example:
A manufacturer of ice cream desires to achieve a target fill weight of 2.50 pounds. Two factors will be examined to determine where they should be set to achieve the target weight. These factors are the filling temperature and the percent overfill. Overfill is the percentage of air incorporated into the ice cream.

Factors and settings are as follows:

Factor	**Description**	+	–
A	Fill temperature, °F	25	20
B	Overfill, %	110	90

Experimental response: Fill weight in pounds

This experiment will be a full 2^2 factorial (two factors at two levels) and will require four runs. No replicates will be run.

The design matrix and responses are shown in the following table.

Run	*A*	*B*	*AB*	Response
1	–	–	+	2.31
2	+	–	–	2.82
3	–	+	–	2.16
4	+	+	+	2.38

Effects are calculated as follows:

***A* main effect:**

$$\left(\frac{2.82+2.38}{2}\right)-\left(\frac{2.31+2.16}{2}\right)=2.60-2.24=+0.36$$

***B* main effect:**

$$\left(\frac{2.16+2.38}{2}\right)-\left(\frac{2.31+2.82}{2}\right)=2.27-2.57=-0.30$$

***AB* interaction effect:**

$$\left(\frac{2.31+2.38}{2}\right)-\left(\frac{2.82+2.16}{2}\right)=2.35-2.49=-0.14$$

The general prediction equation is given by

$$\hat{Y}=\text{Grand average}+\left(\frac{\text{Effect } A}{2}\right)(A)+\left(\frac{\text{Effect } B}{2}\right)(B)+\left(\frac{\text{Effect } AB}{2}\right)(A)(B)$$

where: $\hat{Y}$= the expected value of the response

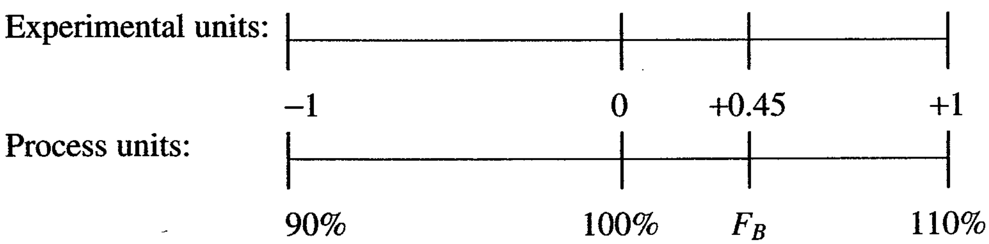
For our specific case where the target value is $y = 2.50$, this expression becomes

$$2.50 = 2.42 + 0.18A - 0.15B - 0.07AB$$

Since we have one prediction equation with two variables, we must fix one. Suppose we set factor A to the + setting, or 25°. The resulting prediction equation becomes

$$\hat{Y} = 2.42 + 0.18(+1) - 0.15(B) - 0.07(+1)(B)$$
$$2.50 = 2.42 + 0.18 - 0.15(B) - 0.07(B)$$
$$2.50 = 2.60 - 0.22(B)$$
$$-0.10 = -0.22(B)$$
$$B = +0.45$$

In order for $B = +0.45$ to be utilized in setting the process factor B (percent overrun), we must translate the coded factor level to a process setting. This can be visualized in the following diagram:



The process setting F_B can be determined by

$$\frac{0.45-0.00}{F_B-100} = \frac{1.0-0.0}{110-100}$$
$$(F_B-100)(1) = 0.45(110-100)$$
$$F_B - 100 = 4.5$$
$$F_B = 104.5\%$$

The proper process conditions to achieve a fill weight of 2.50 pounds are setting the temperature to 25°F and the overfill to 104.5 percent.

AN ALTERNATIVE METHOD FOR EVALUATING EFFECTS BASED ON STANDARD DEVIATION

Montgomery has shown that ln S^2_+/S^2_- has an approximate standard normal distribution. Using the standard deviation as a response and calculating the variance of the average standard deviations where the factor is + and – and then calculating the natural log of the + term divided by the – term yields a test statistic. If the absolute value of this test statistic is greater than $Z_{\alpha/2}$, we may conclude that the effect is statistically significant. We are essentially performing a hypothesis test.

1. *Ho:* $\sigma_+^2 = \sigma_-^2$
   Ha: $\sigma_+^2 \neq \sigma_-^2$
2. Compute the test statistic.

If $\left|\ln\frac{(\bar{S}_+)^2}{(\bar{S}_-)^2}\right| > Z_{\alpha/2}$, reject *Ho* and accept *Ha*. Conclude that the effect is significant.

For a 95 percent level of confidence, $Z_{\alpha/2} = 1.96$.

Example:
Consider the 2^2 experiment where the objective is to minimize the variation in the percent shrinkage of pressure-treated wood. Note that we are not minimizing the shrinkage but rather, the variation in the shrinkage. Two factors will be evaluated. Factor *A* is pressure, where + = 150 psi and – = 100 psi. Factor *B* is time, where + = 4 hours and – = 2 hours. Four experimental runs are planned. Three replicates are run, and the percent shrinkage is determined for each run.

Run	*A*	*B*	*AB*	Responses	Average	Standard deviation
1	–	–	+	2.31, 2.42, 2.46	2.397	0.0777
2	+	–	–	2.46, 2.51, 2.53	2.500	0.0361
3	–	+	–	4.86, 4.77, 4.28	4.637	0.3121
4	+	+	+	3.75, 3.87, 4.61	4.077	0.4658

Effects based on the standard deviation are as follows:

***A* main effect:**

$$\left(\frac{0.0361+0.4658}{2}\right)-\left(\frac{0.0777+0.3121}{2}\right)=0.2510-0.1949=0.0561$$

***B* main effect:**

$$\left(\frac{0.3121+0.4658}{2}\right)-\left(\frac{0.0777+0.361}{2}\right)=0.3890-0.0569=0.3321$$

***AB* interaction:**

$$\left(\frac{0.0777+0.4658}{2}\right)-\left(\frac{0.0361+0.3121}{2}\right)=0.2718-0.1741=0.0977$$

Test for statistical significance as follows:

Effect *A*:

$$\bar{S}_+^2=\left(\frac{0.0361+0.4658}{2}\right)^2=0.0630$$

$$\bar{S}_-^2=\left(\frac{0.0777+0.3121}{2}\right)^2=0.0380$$

$\left|\ln\frac{(\bar{S}_+)^2}{(\bar{S}_-)^2}\right|=\left|\ln\frac{0.0630}{0.0380}\right|=0.51$ $\quad 0.51 \ngtr 1.96$; therefore, factor *A* is not significant.

Effect *B*:

$$\bar{S}_+^2 = \left(\frac{0.3121+0.4658}{2}\right)^2 = 0.1513$$

$$\bar{S}_-^2 = \left(\frac{0.0777+0.0361}{2}\right)^2 = 0.0032$$

$$\left|\ln\frac{(\bar{S}_+)^2}{(\bar{S}_-)^2}\right| = \left|\ln\frac{0.1513}{0.0361}\right| = 3.86 \quad 3.86 > 1.96\text{; therefore, factor } B \text{ is significant.}$$

Effect *AB* interaction:

$$\bar{S}_+^2 = \left(\frac{0.0777+0.4658}{2}\right)^2 = 0.0738$$

$$\bar{S}_-^2 = \left(\frac{0.0361+0.3121}{2}\right)^2 = 0.0303$$

$$\left|\ln\frac{(\bar{S}_+)^2}{(\bar{S}_-)^2}\right| = \left|\ln\frac{0.0738}{0.0303}\right| = 0.89 \quad 0.89 \not> 1.96\text{; therefore, the } AB \text{ interaction is not significant}$$

Conclusion:

To minimize the variation in the actual percent shrinkage, the time of treatment should be two hours. The pressure is not a statistically significant factor with respect to minimizing variation. Note that we reverse the signs of the effect to determine the setting for the factor in order to minimize the response. Since the sign for the main effect for *B* is +, we chose to set factor *B* to the – setting, or two hours.

No calculated effects were made using the shrinkage response; however, it is obvious that setting *B* – would minimize the actual percent shrinkage as well as the variation in the shrinkage.

PLACKETT-BURMAN SCREENING DESIGNS

These series of designs are modestly useful for screening but should be used with great caution. At best, the Plackett-Burman designs are a resolution III. Confounding of main effects with two- and three-factor interactions are significant.

The design columns for several Plackett-Burman designs follow:

$n = 8$ + + + − + − −

$n = 12$ + + − + + + − − − + −

$n = 16$ + + + + − + − + + − − + − − −

$n = 20$ + + − − + + + + − + − + − − − − + + −

$n = 24$ + + + + + − + − + + − − + + − − + − + − − − −

To generate the design matrix, the appropriate generating column is selected. The first column is generated by placing the signs in a vertical column. The last sign of the first column becomes the first sign in the second column, and the remainder of the second column follows the sequence designated by the generating column. After $n - 1$ columns have been developed, the final row is made by using all –'s.

An example is shown for a seven-factor, eight-run Plackett-Burman design.

Run	*A*	*B*	*C*	*D*	*E*	*F*	*G*
1	+	–	–	+	–	+	+
2	+	+	–	–	+	–	+
3	+	+	+	–	–	+	–
4	–	+	+	+	–	–	+
5	+	–	+	+	+	–	–
6	–	+	–	+	+	+	–
7	–	–	+	–	+	+	+
8	–	–	–	–	–	–	–

In the event that less than eight factors are to be screened, simply drop the required number of columns (factors).

Case study:

You have been asked to examine a process with respect to three factors that are suspected to influence a response. The objective is for the process to yield a response of 55.0 with a minimum of variation. The factors and their settings for this experiment are as follows:

The design matrix and data are as follows:

		Setting	
Factor	Description	–	+
A	Feed rate, lbs/hr.	40	60
B	Temperature, °C	30	70
C	Percent filler	22	30

Run	*A*	*B*	*C*	*AB*	*AC*	*BC*	*ABC*	I	II	Average	Standard deviation
1	–	–	–	+	+	+	–	40.48	45.52	43.0	3.56
2	+	–	–	–	–	+	+	30.01	27.99	29.0	1.43
3	–	+	–	–	+	–	+	50.92	67.08	59.0	5.71
4	+	+	–	+	–	–	–	42.05	31.95	37.0	7.14
5	–	–	+	+	–	–	+	69.69	72.31	71.0	1.85
6	+	–	+	–	+	–	–	47.95	50.05	49.0	1.48
7	–	+	+	–	–	+	–	69.54	64.46	67.0	3.59
8	+	+	+	+	+	+	+	42.46	47.54	45.0	9.98

Effects based on average response are as follows:

Main effect *A*:

$$\left(\frac{29+37+49+45}{4}\right)-\left(\frac{43+59+71+67}{4}\right)=-10$$

Main effect *C*:

$$\left(\frac{71+49+67+45}{4}\right)-\left(\frac{43+29+59+37}{4}\right)=+16$$

Interaction effect *AC*:

$$\left(\frac{43+59+49+45}{4}\right)-\left(\frac{29+37+71+67}{4}\right)=-2$$

Interaction *ABC*:

$$\left(\frac{29+59+71+45}{4}\right)-\left(\frac{43+37+49+67}{4}\right)=2$$

Main effect *B*:

$$\left(\frac{59+37+67+45}{4}\right)-\left(\frac{43+29+71+49}{4}\right)=+4$$

Interaction effect *AB*:

$$\left(\frac{43+37+71+45}{4}\right)-\left(\frac{29+59+49+67}{4}\right)=-2$$

Interaction *BC*:

$$\left(\frac{43+29+67+45}{4}\right)-\left(\frac{59+37+71+49}{4}\right)=-8$$

Determine the statistical significance for effects based on average responses. The level of confidence is chosen to be 95 percent. The error for the calculated effects is given by

$$\text{Error, } E=t_{\alpha/2,(r-1)2^{k-f}}\left(Sp\sqrt{\frac{2}{pr}}\right)$$

where: $\alpha = 0.05$
$r = 2$
$f = 0$
$p = 4$

$$Sp = \sqrt{\frac{(3.56)^2 + (1.43)^2 + \ldots + (9.98)^2}{8}} = 5.20 \qquad \sqrt{\frac{2}{(4)(2)}} = 0.50$$

$$t_{0.025,8} = 2.303$$

$$\text{Error, } E = (2.303)(5.20)(0.50) = 5.99$$

Any effects greater than the experimental error 5.99 are deemed statistically significant at the 95 percent level. The effects that are significant are main effects *A*, *C*, and the two-factor interaction effect *BC*.

The model for predicting the response using only those effects that are significant is

$$Y_{\text{expected}} = \text{grand average} - \tfrac{1}{2}(\text{effect } A)(A) + \tfrac{1}{2}(\text{effect } C)(C) - \tfrac{1}{2}(\text{effect } BC)(B)(C)$$

$$Y_{\text{expected}} = 50 - 5(A) + 8(C) - 4(B)(C)$$

Effects based on standard deviation are as follows:

Main effect *A*:

$$\left(\frac{1.43 + 7.14 + 1.48 + 9.98}{4}\right) - \left(\frac{3.56 + 5.71 + 1.85 + 3.59}{4}\right) = 0.62$$

Main effect *B*:

$$\left(\frac{5.71 + 7.14 + 3.59 + 9.98}{4}\right) - \left(\frac{3.56 + 1.43 + 1.85 + 1.48}{4}\right) = 2.31$$

Main effect *C*:

$$\left(\frac{1.85 + 1.48 + 3.59 + 9.98}{4}\right) - \left(\frac{3.56 + 1.43 + 5.71 + 7.14}{4}\right) = 0.11$$

Interaction *AB* effect:

$$\left(\frac{3.56 + 7.14 + 1.85 + 9.98}{4}\right) - \left(\frac{1.43 + 5.71 + 1.48 + 3.59}{4}\right) = 1.23$$

Interaction *AC* effect:

$$\left(\frac{3.56 + 5.71 + 1.48 + 9.98}{4}\right) - \left(\frac{1.43 + 7.14 + 1.85 + 3.59}{4}\right) = 0.78$$

Interaction *BC* effect:

$$\left(\frac{3.56 + 1.43 + 3.59 + 9.98}{4}\right) - \left(\frac{5.71 + 7.14 + 1.85 + 1.48}{4}\right) = 0.27$$

Interaction *ABC* effect:

$$\left(\frac{1.43+5.71+1.85+9.98}{4}\right)-\left(\frac{3.56+7.14+1.48+3.59}{4}\right)=0.37$$

Determine the statistical significance for effects based on standard deviation as a response. Each one of the effects based on standard deviation will be tested independently for statistical significance. The level of significance chosen is 90 percent.

The test statistic will be the natural log of the ratio of the square of the average standard deviation of the –'s to the +'s.

$$\text{Test statistic: } \left|\ln\frac{(\bar{S}_+)^2}{(\bar{S}_-)^2}\right|$$

If the test is greater than $Z_{\alpha/2}$, we will reject the hypothesis that the effect is not statistically significant at a level of confidence of $1 - \alpha$. In this case, the test statistic must be greater than 1.645.

Calculate the test statistic for the effects as follows:

Main effect *A*:

$$(\bar{S}_+)^2=\left(\frac{1.43+7.14+1.48+9.98}{4}\right)^2=5.01$$

$$(\bar{S}_-)^2=\left(\frac{3.56+5.71+1.85+3.59}{4}\right)^2=3.68$$

$$\left|\ln\frac{(\bar{S}_+)^2}{(\bar{S}_-)^2}\right|=\left|\ln\frac{5.01}{3.68}\right|=\left|\ln\frac{25.10}{13.54}\right|=\left|\ln 1.85\right|=0.62$$

Since the test statistic is less than the critical value of 1.645, we cannot reject the null hypothesis that the variation from a positive setting of *A* is any different than the variation from a negative setting of *A*. We conclude that factor *A* is not a significant factor affecting variation.

Similar calculations for all of the factors and interactions give the following test statistics:

Factor	Test statistic	
A	0.62	
B	2.31	(Statistically significant at a confidence level of 90%)
C	0.11	
AB	1.23	
AC	0.78	
BC	0.27	
ABC	0.37	

In order to minimize variation, factor *B* must be set at the – setting, or 30°C.

ACHIEVING THE FINAL OBJECTIVE

The target value for the primary response is to obtain a value of 55.0 with a minimum of variation. Minimum variation is obtained with factor B set at the – setting, or 30° C. The expected value for Y now becomes

$$Y_{\text{expected}} = 50 - 5(A) + 8(C) - 4(B)(C) \text{ since } B = -1$$

$$Y_{\text{expected}} = 50 - 5(A) + 8(C) - 4(-1)(C)$$

$$Y_{\text{expected}} = 50 - 5(A) + 12(C)$$

Setting $Y = 55$, we may adjust factor A or C to achieve the desired target value. Of the two factors, factor A had more of an affect on variation, even though it was not statistically significant. We will set factor A to its negative setting (40 lb./hr. feed rate).

This leaves only factor C to adjust to target.

$$Y_{\text{expected}} = 55 - (5)(-1) + 12(C)$$

$$Y_{\text{expected}} = 60 + 12(C)$$

$$55 = 60 + 12(C)$$

$$C = -0.42$$

This setting is in standardized units of the experimental design, where –1 = 22%, 0 = 26%, and +1 = 30%.

Experimental units:	–1	–0.42	0	+1
Process units:	22%	24.4%	26%	30%

Setting the percent filler to 24.4 percent will achieve a desired response of 55.0.

BIBLIOGRAPHY

Barrentine, L. B. 1999. *An Introduction to Design of Experiments.* Milwaukee, WI: ASQ Quality Press.

Box, G. E. P., W. G. Hunter, and J. S. Hunter. 1978. *Statistics for Experimenters.* New York: John Wiley and Sons.

Montgomery, D. C. 1997. *Design and Analysis of Experiments.* 4th edition. New York: John Wiley and Sons.

Schmidt, S. R., and R. G. Launsby. 1994. *Understanding Industrial Designed Experiments.* 4th edition. Colorado Springs, CO: Air Academy Press.